High Relative Degree Control Barrier Functions Under Input Constraints

60th IEEE Conference on Decision and Control
Austin, Texas, United States, December 17th 2021

Joseph Breeden, Dimitra Panagou

Department of Aerospace Engineering

University of Michigan, Ann Arbor, MI, USA

UNIVERSITY OF MICHIGAN

Preliminaries

- System: $\dot{x} = f(x) + g(x)u$, where $x \in \mathbb{R}^n, u \in U \subset \mathbb{R}^m$ where U is compact
- Safe set: $S = \{x \in \mathbb{R}^n \mid h(x) \leq 0\}$ for $h : \mathbb{R}^n \to \mathbb{R}$
  - Goal is to render trajectories always inside S
- Assume f, g, h are r-times continuously differentiable, where r is the relative-degree of h (lowest r such that $h^{(r)}$ depends on u)

- Control Barrier Functions (CBFs) are a method to certify existence of safe control inputs
- S is rendered forward invariant if and only if $\dot{h}(x,u) = \frac{\partial h(x)}{\partial x}\dot{x} \leq 0$ for all $x \in \partial S$ (Nagumo's Theorem)
  - In practice, enforce $\dot{h}(x,u) \leq \alpha(-h(x))$ for all $x \in S$

Definition 1

A continuously differentiable function $h : \mathbb{R}^n \to \mathbb{R}$ is a Control Barrier Function (CBF) on set S for control set U if there exists $\alpha \in \mathcal{K}$ such that

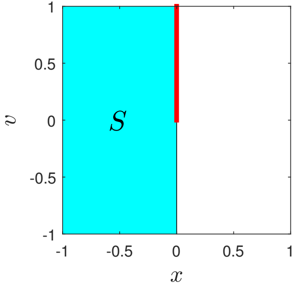
$$\inf_{u \in U} [\dot{h}(x,u) - \alpha(-h(x))] \leq 0, \ \forall x \in S \, .$$

Example

- Double-integrator with a position constraint

$$\dot{x} = v$$
$$\dot{v} = u$$

$$h([x,\ v]^{\mathrm{T}}) = x$$
$$S = \{(x, v) \in \mathbb{R}^2 \mid x \leq 0\}$$
$$\dot{h}([x,\ v]^{\mathrm{T}}) = v \to \exists (x, v) \in \partial S : \dot{h}([x,\ v]^{\mathrm{T}}) \not\leq 0$$



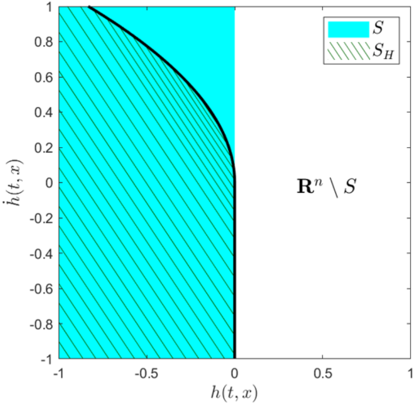
- h is not a CBF
- $\dot{h}$ does not depend on u, so h is of “high-relative-degree” (i.e. $r > 1$)

High Relative Degree Constraints

- Fact: $H = h + \dot{h}$ is a CBF for the prior example, provided no input constraints (i.e. $U = \mathbb{R}^m$)
- Prior work on converting high-relative-degree h to CBFs
  - Backstepping approach (Hsu, Xu, Ames, ACC 2015)
  - Exponential CBFs (Nguyen, Sreenath, ACC 2016)
  - Compositions with $h^{(r)}$ (Ames, Xu, Grizzle, Tabuada, TAC 2017)
  - Backup Controllers (Squires, Pierpaoli, Egerstedt, CCTA 2018)
  - Higher Order CBFs (Xiao, Belta, CDC 2019)

Contribution

- Suggest two forms of $H : \mathbb{R}^n \to \mathbb{R}$ that are CBFs in the presence of input constraints, where $H(x) \geq h(x)$ for all $x \in \mathbb{R}^n$ so that $S_H = \{x \in \mathbb{R}^n \mid H(x) \leq 0\} \subset S$
- $S_H =$ the "inner safe set" $=$ set of allowable initial conditions
- Existence of a CBF implies we can render S_H forward invariant



Method 1

- For some policy $u^*: \mathbb{R}^n \to U$, define $\psi_x(t; x, u^*) = y(t)$ according to the initial value problem

$$\dot{y} = f(y) + g(y)u^*(y),\ y(0) = x$$

  and $\psi_h(t; x, u^*) = h(\psi_x(t; x, u^*))$
- E.g. $u^*_{\text{ball}}(x) = \arg\min_{u \in U} h^{(r)}(x, u) = \arg\min_{u \in U} L_g L_f^{r-1} h(x) u$
- u^* called the "nominal evading maneuver" in [Squires, Pierpaoli, Egerstedt, CCTA 2018]
- We do not need closed-form expressions for ψ_x, ψ_h

Method 1

- Define $H(x) \triangleq \sup_{t\geq 0} \psi_h(t; x, u^*)$

Assumption 1

Assume H exists and is differentiable everywhere in S.

Theorem 1

H is a CBF on the set S_H for the control set U, provided S_H is nonempty.

- We do not need closed-form expressions for H
- The CBF condition $\dot{H}(x, u) \leq \alpha(-H(x))$ is still control-affine

Method 2

- Let $u' : \mathbb{R}^n \to U$ be a policy such that

$$h^{(r)}(x, u'(x)) = -a_{max}, \forall x \in S$$

  for some fixed $a_{max} \in \mathbb{R}_{>0}$ (provided a_{max} exists).
- One such a_{max} is

$$a_{max} \triangleq \max \left(\left\{ a \in \mathbb{R} \;\middle|\; \forall x \in S, \exists v \in (L_g L_f^{r-1} h(x))^{\perp} : -\frac{(a + L_f^r h(x))(L_g L_f^{r-1} h(x))}{||L_g L_f^{r-1} h(x)||^2} + v \in U \right\} \right)$$

- $\psi_h(t; x, u')$ is a polynomial in t

$$\psi_h(t; x, u') = \sum_{i=0}^{r-1} \frac{1}{i!} h^{(i)}(x) t^i - \frac{1}{r!} a_{max} t^r$$

Method 2

- Define $H'(x) \triangleq \sup_{t\geq 0} \psi_h(t; x, u')$
- Existence and differentiability of H' are guaranteed

Theorem 2

H' is a CBF on the set $S_{H'}$ for the control set U, provided $S_{H'}$ is nonempty.

Complexity

- Method 1 requires propagating a $n \times 1$ and a $n \times n$ ordinary differential equation
- Method 2 requires finding the roots of a $(r-1)$-dimensional polynomial

Application 1

- Double integrator with a spherical exclusion region

$$\dot{x} = \begin{bmatrix} \dot{r} \\ \dot{v} \end{bmatrix} = \begin{bmatrix} v \\ u \end{bmatrix}$$

$$r, v, \in \mathbb{R}^3, \; u \in U = \{u \in \mathbb{R}^3 \mid \|u\|_\infty \leq u_{max}\}$$

- $h(x) = \rho - \|r - r_s\|$ for fixed $r_s \in \mathbb{R}^3$
- $a_{max} = u_{max}$

Application 1

- Lyapunov function

$$V(x) = \frac{1}{2}||r - r_p||^2 + \frac{1}{2}k_2||v - k_1(r - r_p)||^2$$

where $r_p \in \mathbb{R}^3$ is a target location.

- Control law:

$$u(x) = \underset{u \in U, \delta \in \mathbb{R}}{\arg\min}\, u^{\mathrm{T}} u + J\delta^2 \quad \text{such that}$$

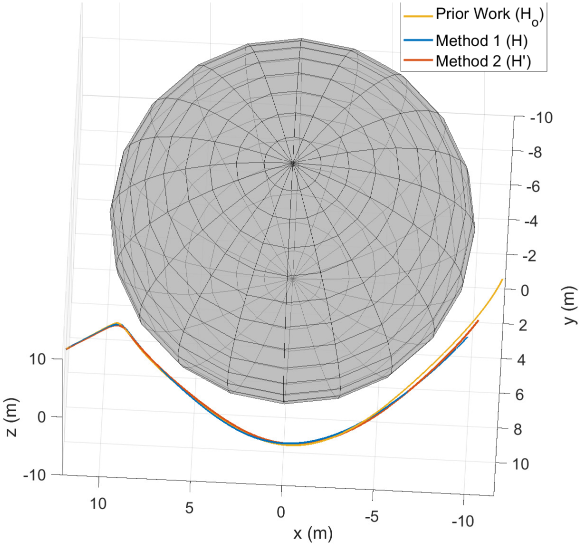
$$L_f H(x) + L_g H(x) u \leq \alpha(-H(x))$$

$$L_f V(x) + L_g V(x) u + \delta \leq -k_3 V(x)$$

Application 1 Results

- Comparison CBF (no guarantee of input constraint satisfaction) from [Ames, Xu, Grizzle, Tabuada, TAC 2017]

$$H_o(x) = \left(\arctan(\dot{h}(x)) + \frac{\pi}{2}\right) h(x)$$



Application 1 Results

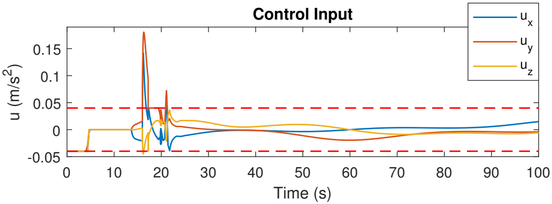


Figure: The control input using $H_o(x)$ from prior work, which necessitates using control inputs outside the prescribed bounds (dashed red lines) for the QP to have a solution

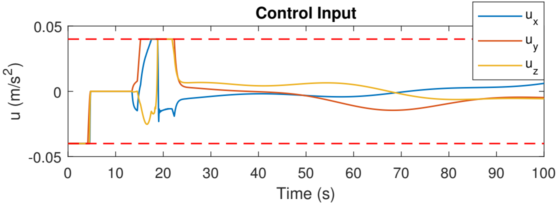


Figure: The control input using $H(x)$ as in Method 1

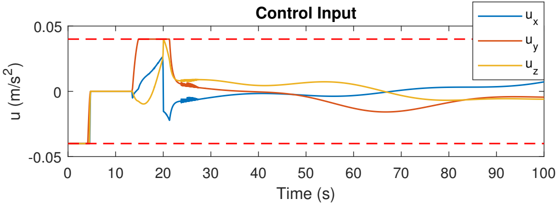


Figure: The control input using $H'(x)$ as in Method 2

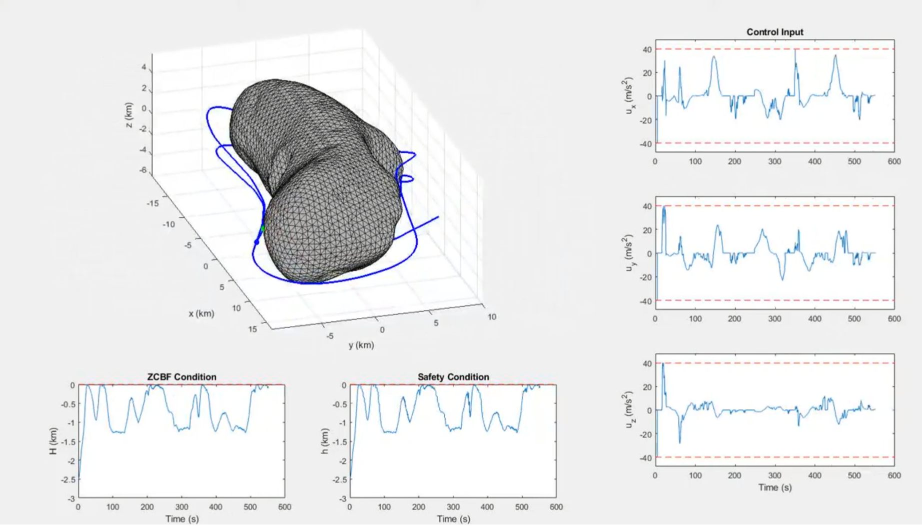
Application 2 (Spacecraft)

- Dynamics

$$\dot{x} = \begin{bmatrix} \dot{r} \\ \dot{v} \end{bmatrix} = \begin{bmatrix} v \\ f_\mu(r) + u \end{bmatrix}$$

- Simulated using only H' to reduce computations
- $a_{max} < u_{max}$ ($a_{max} \approx \frac{1}{2} u_{max}$ in this simulation)
- Collection of CBFs $\{H_i'(x)\}_{i=1}^{7790}$ for point cloud model $\{r_{s,i}\}_{i=1}^{7790}$
- $r_p = r_p(x)$ moving target (for Lyapunov function)

Application 2 Results



https://youtu.be/JKj3PUrYnEg

Conclusion

- Presented two explicit methods for constructing CBFs with input constraints
- Feasibility of $\dot{H} \leq \alpha(-H)$ under input constraints is guaranteed within the zero sublevel sets of both CBFs
- Expanded utility of CBFs as an online control methodology
- Current/future work
  - Input constraints + disturbances + sampled-data dynamics
  - Fuel-optimality/planning
  - Multi-agent space systems

Acknowledgements

The authors would like to acknowledge the support of the U.S. National Science Foundation

High Relative Degree Control Barrier Functions Under Input Constraints

60th IEEE Conference on Decision and Control
Austin, Texas, United States, December 17th 2021

Joseph Breeden, Dimitra Panagou

Department of Aerospace Engineering

University of Michigan, Ann Arbor, MI, USA

UNIVERSITY OF
MICHIGAN